

DISTORTION OF IMBEDDINGS OF GROUPS OF INTERMEDIATE GROWTH INTO METRIC SPACES

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ABSTRACT. For every metric space \mathcal{X} in which there exists a sequence of finite groups of bounded-size generating set that does not embed coarsely, and for every unbounded, increasing function ρ , we produce a group of subexponential word growth all of whose imbeddings in \mathcal{X} have distortion worse than ρ .

This applies in particular to any B-convex Banach space \mathcal{X} , such as Hilbert space.

1. INTRODUCTION

Let G be a finitely generated group, and let (\mathcal{X}, d) be a metric space. The extent to which G , with its word metric, may be imbedded in \mathcal{X} with not-too-distorted metric is an asymptotic invariant of G , introduced by Gromov in [9, §7.E]. The general definition of distortion is as follows:

Definition 1.1. Consider a 1-Lipschitz map $\Phi : (\mathcal{Y}, d) \rightarrow (\mathcal{X}, d)$ between metric spaces. Its *distortion* is the function

$$\rho_\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad \rho_\Phi(t) = \inf_{d(y, y') \geq t} d(\Phi(y), \Phi(y')).$$

It is the largest increasing function $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\rho(d(y, y')) \leq d(\Phi(y), \Phi(y')) \leq d(y, y') \text{ for all } y \neq y' \in \mathcal{Y}.$$

If \mathcal{Y} has bounded diameter, then $\rho_\Phi(t) = +\infty$ for all $t > \text{diam } \mathcal{Y}$. We say that Φ has distortion *better than* ρ if $\rho_\Phi(t) > \rho(t)$ for all $t \in \mathbb{R}_+$ large enough, *worse than* ρ if $\rho_\Phi(t) < \rho(t)$ for all $t \in \mathbb{R}_+$ large enough, and that Φ is a *coarse imbedding* if its distortion is unbounded¹.

More generally, if $(\mathcal{Y}_i)_{i \in I}$ is a family of metric spaces, a *coarse imbedding* is a sequence $(\Phi_i : \mathcal{Y}_i \rightarrow \mathcal{X})$ of 1-Lipschitz imbeddings with $\inf_{i \in I} \rho_{\Phi_i}(t)$ an unbounded function of t .

Our main result is:

Theorem A (= Theorem 7.2). *Let \mathcal{X} be a metric space, and let $(G_i)_{i \in \mathbb{N}}$ be a sequence of d -generated finite groups that do not imbed coarsely in \mathcal{X} .*

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¹An alternative terminology is *uniform imbedding*, which we avoid.

Then, for every unbounded increasing function $\rho: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, there exists a finitely generated group W of subexponential growth such that every imbedding of W in \mathcal{X} has distortion worse than ρ .

Furthermore, the group W contains an infinite subsequence of the G_i 's.

Corollary B (= Corollary 8.4). *For every unbounded increasing function $\rho: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and for every B-convex Banach space \mathcal{X} (e.g. Hilbert space; see §8 for the definition) there exists a finitely generated group W of subexponential word growth such that every imbedding of W in \mathcal{X} has distortion worse than ρ .*

Furthermore, the group W depends on ρ but only mildly on \mathcal{X} : for every unbounded increasing function ρ there exists a group W of subexponential growth, such that every imbedding of W in a B-convex Banach space \mathcal{X} has distortion worse than $c\rho$ for some constant c ; and moreover c depends only on the convexity parameters n, ϵ of \mathcal{X} , see Remark 8.5.

Recall that a finitely generated group G has *subexponential growth* if, for every $\lambda > 1$, the number of group elements that are products of at most n generators grows more slowly than the exponential function λ^n . A group has *locally subexponential growth* if all its finitely generated subgroups have subexponential growth.

Arzhantseva, Druţu and Sapir construct in [1, Theorem 1.5], for every unbounded increasing function ρ , a group which imbeds coarsely into Hilbert space, and such that all of its imbeddings into a uniformly convex Banach space have distortion worse than ρ (for a slightly weaker definition of “worse” than ours). Olshansky and Osin construct, moreover, amenable groups with the same property in [14, Corollary 1.4].

These examples all have exponential growth. In contrast, the main point of our construction is to produce such groups having subexponential growth. These are in particular the first known examples of groups whose simple random walks have trivial Poisson boundary and with arbitrarily bad distortion in every imbedding into Hilbert space.

It follows from [13, Theorem 1.1] by Naor and Peres that, if an amenable group G admits an imbedding with distortion better than $n^{1/2+\epsilon}$ for some $\epsilon > 0$, then every simple random walk on G has trivial Poisson boundary. Our result shows that groups with trivial Poisson boundary for every simple random walk may have arbitrarily bad distortion in every imbedding into Hilbert space.

Note that groups of subexponential growth are amenable, and Bekka, Ch  rix and Valette show in [5] that amenable groups imbed coarsely into Hilbert space. In fact, their imbeddings can be shown to have distortion better than an unbounded function which depends only on the F  lner function. Tessera [15, Theorem 10] gives such formul  e in terms of the isoperimetric profile inside balls.

For example, consider the Grigorchuk groups G_ω of intermediate growth, introduced in [8]. They admit “self-similar random walks” μ_ω in the sense of [3, §6] and [10], with asymptotic entropy $H(\mu_\omega^n) \lesssim n^{1/2}$ by [3, Corollary 6.3]; so their probability of return satisfies $\mu_\omega^n(1) \gtrsim \exp(-n^{1/2})$ using the general estimate $\mu^n(1) \geq \exp(-2H(\mu^n))$, their F  lner function satisfies $F(n) \lesssim \exp(n^2)$ using Nash inequalities (see [16, Corollary 14.5(b)]), and therefore they admit imbeddings into Hilbert space of distortion better than $t^{1/2-\epsilon}$ for every $\epsilon > 0$, by a result of Gournay [7].

On the other hand, groups of subexponential growth can have arbitrarily large Følner function [6]. Our result can therefore be seen as a strengthening of this fact.

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2. SKETCH OF THE PROOF

We apply the construction of [4], which constructs, for any countable group B all of whose finitely generated subgroups have subexponential growth, a finitely generated group of subexponential growth in which B imbeds; see §3 and §4. We take for B a restricted direct product of finite groups H_i with poor imbedding properties, following the idea of Arzhantseva, Druţu and Sapir [1].

We proceed in three steps: first, given a sequence of finite groups G_1, G_2, \dots with bounded number of generators, we imbed each G_i as the derived subgroup of a finite group H_i with one more generator, in such a manner that the metric on G_i is at universally bounded distance from a particular metric on $[H_i, H_i]$, the *perfect metric* (see §5).

We then show in §6 that an arbitrary subset of the H_i 's imbeds in a finitely generated group W of subexponential growth with controlled distortion; more precisely, the distortion constants of an imbedded H_i in W depends only on the previous H_j 's, but not on H_i .

Finally, in §7 we assume that the H_i do not imbed coarsely in a metric space \mathcal{X} . Given any unbounded increasing function ρ , we select the subset of H_i 's appropriately so that the distortion of W in \mathcal{X} is worse than ρ .

To prove the corollary in §8, we choose for the G_i , or even directly for the H_i , a family of superexpanders such as those constructed by Lafforgue in [12].

3. PERMUTATIONAL WREATH PRODUCTS

We recall briefly some notions introduced in [4]. For details, we refer to that article.

Let $G = \langle S \rangle$ be a finitely generated group acting on the right on a set X . We consider X as the vertex set of a graph still denoted X , with for all $x \in X, s \in S$ an edge labelled s from x to xs . We denote by d the path metric on this graph.

Definition 3.1. A sequence (x_0, x_1, \dots) in X is *spreading* if for all R there exists N such that if $i, j \geq N$ and $i \neq j$ then $d(x_i, x_j) \geq R$.

Definition 3.2. A sequence (x_i) in X *locally stabilises* if for all R there exists N such that if $i, j \geq N$ then the S -labelled radius- R balls centered at x_i and x_j in X are equal.

Definition 3.3. A sequence of points (x_i) in X is *rectifiable* if for all i, j there exists $g \in G$ with $x_i g = x_j$ and $x_k g \neq x_\ell$ for all $k \notin \{i, \ell\}$.

Definition 3.4. The group G acting on X has the *subexponential wreathing property* if for any finitely generated group of subexponential growth H the restricted wreath product $H \wr_X G$ has subexponential growth.

We summarize, in the following proposition, an example of group G , action on X and rectifiable, separating, locally stabilizing sequence (x_i) with the subexponential wreathing property:

Proposition 3.5 ([4, Lemma 4.9] and [2, Proposition 4.4]). *Let $G = G_{012} = \langle a, b, c, d \rangle$ denote the first Grigorchuk group. Recall that it acts on set of infinite sequences $\{0, 1\}^\infty$ over a two-letter alphabet, which is naturally the boundary of a binary rooted tree. Denote by $X = \mathbf{1}^\infty G_{012}$ the orbit of the rightmost ray.*

Then G has the subexponential wreathing property in its action on X , and the sequence (x_i) defined by $x_i = \mathbf{0}^i \mathbf{1}^\infty$ is rectifiable, spreading and locally stabilizing. \square

4. IMBEDDING GROUPS OF LOCALLY SUBEXPONENTIAL GROWTH IN W

We assume that a group G acting on a set X , and a separating, spreading, locally stabilizing sequence (x_i) of elements of X have been fixed; such data exist by Proposition 3.5. We repeat, in this Section, the contents of [4, §6], since they are fundamental to the argument.

Let B be a group, and let (b_1, b_2, \dots) be a sequence in B . We will construct a rapidly increasing sequence $0 \leq n(1) < n(2) < \dots$ later; assuming this sequence given, we define $f: X \rightarrow B$ by

$$f(x_{n(1)}) = b_1, \quad f(x_{n(2)}) = b_2, \quad \dots, \quad f(x) = 1 \text{ for other } x.$$

We then consider the subgroup $W = \langle G, f \rangle$ of the unrestricted wreath product $B^X \rtimes G$.

Lemma 4.1 ([4, Lemma 6.1]). *Denote by B_0 the subgroup of B generated by $\{b_1, b_2, \dots\}$. If the sequence (x_i) is separating, then W contains $[B_0, B_0]$ as a subgroup.*

Proof. Without loss of generality and to lighten notation, we rename B_0 into B . We also denote by $\iota: B \rightarrow B^X \rtimes G$ the imbedding of B mapping the element $b \in B$ to the function $X \rightarrow B$ with value b at x_0 and 1 elsewhere. We shall show that W contains $\iota([B, B])$. For this, denote by H the subgroup $\iota(B) \cap W$.

We first consider an elementary commutator $g = [b_i, b_j]$. Let $g_i, g_j \in G$ respectively map x_i, x_j to x_0 , and be such that $g_i g_j^{-1}$ maps no x_k to x_ℓ with $k \neq \ell$, except for $x_i g_i g_j^{-1} = x_j$. Consider $[f^{g_i}, f^{g_j}] \in W$; it belongs to B^X , and has value $[b_i, b_j]$ at x_0 and is trivial elsewhere, so equals $\iota(g)$ and therefore $\iota(g) \in H$.

We next show that H is normal in B^X . For this, consider $h \in H$. It suffices to show that $h^{\iota(b_i)}$ belongs to H for all i . Now $h^{\iota(b_i)} = h^{f^{g_i}}$ belongs to H , and we are done. \square

Proposition 4.2 ([4, Proposition 6.2]). *Let G be a group acting on X . Let the sequence (x_i) in X be spreading and locally stabilizing. Let a sequence of elements (b_i) be given in the group B , all of the same order $\in \mathbb{N} \cup \{\infty\}$.*

For all $i \in \mathbb{N}$, let f_i be the finitely supported function $X \rightarrow B$ with $f_i(x_{n(j)}) = b_j$ for all $j \leq i$, all other values being trivial, and denote by W_i the group $\langle f_i, G \rangle$.

Then for every increasing sequence $(m(i))$ there is a choice of $(n(i))$ such that the ball of radius $m(i)$ in W coincides with the ball of radius $m(i)$ in W_i , via the identification $f \leftrightarrow f_i$.

Furthermore, the term $n(i)$ depends only on $m(i)$ and on the ball of radius $m(i)$ in $\langle b_1, \dots, b_{i-1} \rangle$.

Proof. Choose $n(i)$ such that $d(x_j, x_k) \geq m(i)$ for all $j \neq k$ with $k \geq n(i)$, and such that the balls of radius $m(i)$ around $x_{n(i)}$ and x_j coincide for all $j > n(i)$.

Consider then an element $h \in W$ in the ball of radius $m(i)$, and write it in the form $h = (c, g)$ with $c: X \rightarrow B$ and $g \in G$. The function c is a product of conjugates of f by words of length $< R$. Its support is therefore contained in the union of balls of radius $m(i) - 1$ around the x_j , with j either $\geq n(i)$ or of the form $n(k)$ for $k < i$. In particular, the entries of c are in $\langle b_1, \dots, b_{i-1} \rangle \cup \bigcup_{j \geq i} \langle b_j \rangle$. For $j > n(i)$, the restriction of c to the ball around x_j is determined by the restriction of c to the ball around $x_{n(i)}$, via the identification $b_i \mapsto b_j$, because the neighbourhoods in X coincide and all cyclic groups $\langle b_j \rangle$ are isomorphic.

It follows that the element $h \in W$ is uniquely determined by the corresponding element in W_i . \square

Corollary 4.3 ([4, Corollary 6.3]). *Let G be a group acting on X with the subexponential wreathing property. Let the sequence (x_i) be spreading and locally stabilizing.*

If B has locally subexponential growth, then W has subexponential growth.

Proof. Let $Z = \langle z \rangle$ be a cyclic group whose order (possibly ∞) is divisible by the order of the b_i 's. We replace B by $B \times Z$ and each b_i by $b_i z$, so as to guarantee that all generators in B have the same order.

Let ϵ_i be a decreasing sequence tending to 1. Denote by v_i the growth function of the group W_i introduced in Proposition 4.2, and by w the growth function of W . Let $m(i)$ be such that

$$v_i(m(i)) \leq \epsilon_i^{m(i)}.$$

Such an $m(i)$ exists, because $B \wr_X G$ has locally subexponential growth. Since the balls of radius $m(i)$ coincide in W and W_i , we also have $w(m(i)) \leq \epsilon_i^{m(i)}$. Then, if $R > m(i)$, we get

$$w(R) \leq \epsilon_i^{R+m(i)},$$

so $\lim \sqrt[R]{w(R)} \leq \epsilon_i$. Since this holds for all i , the growth of W is subexponential. \square

5. IMBEDDING A GROUP IN A DERIVED SUBGROUP

Let $G = \langle S \rangle$ be a group with fixed generating set S . We denote by $\|\cdot\|_S$ the word norm on G , or $\|\cdot\|_G$ if the generating set is clear. Let us now define another norm on $[G, G]$. For this, let us say that a word w in the free group F_S is *balanced* if it belongs to $[F_S, F_S]$; namely, if it contains as many s 's as s^{-1} 's for every letter $s \in S$. The *perfect norm* on $[G, G]$ is

$$\|g\|_{\text{perfect}} = \min\{\|w\| : w \in F_S \text{ is a perfect word representing } g\}.$$

We denote by $d_S(x, y) = d_G(x, y) = \|xy^{-1}\|_S$ and $d_{\text{perfect}}(x, y) = \|xy^{-1}\|_{\text{perfect}}$ the corresponding distances.

Proposition 5.1. *Let $G = \langle S \rangle$ be a finite group with fixed generating set of cardinality d . Then there exists a finite group $H = \langle T \rangle$ with generating set of cardinality $d + 1$ and an imbedding $\iota: G \rightarrow [H, H]$ such that*

$$2\|g\|_G \leq \|\iota(g)\|_{\text{perfect}} \leq 4\|g\|_G \text{ for all } g \in G.$$

Proof. Let m denote the cardinality of G . Since $G * \mathbb{Z}$ is residually finite, there exists a finite quotient $Q = \langle S \cup \{x\} \rangle$ of $G * \mathbb{Z}$ such that the balls of radius m in $G * \mathbb{Z}$ and in Q coincide.

In the group $Q \wr C_{2m}$, consider the following elements: for every $s \in S$, the function $t_s: C_{2m} \rightarrow Q$ with values $(1, s, s^2, \dots, s^{m-1}, 1, s^x, s^{2x}, \dots, s^{(m-1)x})$; and the generator r of C_{2m} . Set

$$T = \{t_s : s \in S\} \cup \{r\} \text{ and } H = \langle T \rangle.$$

Define $\iota: G \rightarrow Q \wr C_{2m}$ by

$$\iota(g): C_{2m} \rightarrow Q \text{ taking values } (g, \dots, g, g^x, \dots, g^x).$$

Note first that, for $s \in S$, we have $\iota(t) = [t_s, r]$. This immediately implies $\iota(G) \leq [H, H]$ and gives the inequality $\|\iota(h)\|_{\text{perfect}} \leq 4\|h\|_G$.

Note then that if a word of length $\leq m/2$ in T is not of the form $q_0 r^{-1} q_1 r \cdots q_{2n-2} r^{-1} q_{2n-1} r q_{2n}$ with all $q_i \in \langle t_s : s \in S \rangle$, then it cannot belong to the image of ι . On the other hand, if it is of that form, then write it as $f: C_{2m} \rightarrow Q$, and note that $f(1)$ depends only on q_1, \dots, q_{2n-1} and has length at most the sum of their lengths. This gives the other inequality $2\|h\|_G \leq \|\iota(h)\|_{\text{perfect}}$. \square

6. IMBEDDING A SEQUENCE OF GROUPS IN W

We now apply the construction of Section 4 to a restricted direct product of finite groups. The heart of the argument is the following variant of Corollary 4.3:

Proposition 6.1. *Let $(H_i)_{i \in \mathbb{N}}$ be a sequence of d -generated finite groups.*

Then there exists a family of groups $(W_S)_{S \subset \mathbb{N}}$ indexed by subsets S of \mathbb{N} , each of subexponential growth, with the following property: for all $s \in S$, there is an imbedding $\Psi_s: [H_s, H_s] \rightarrow W_S$ that is (K, L) -bi-Lipschitz with respect to the perfect metric of $[H_s, H_s]$ and the word metric on W_S , and such that the constants K, L depend only on $\{H_i: i < s\}$.

Furthermore, if $S = \{s(1), s(2), \dots\}$, then $W_{\{s(1), \dots, s(i+1)\}}$ is constructed out of $W_{\{s(1), \dots, s(i)\}}$, and the sequence $(W_{\{s(1), \dots, s(i)\}})_{i \in \mathbb{N}}$ converges to W_S in the Cayley topology.

Proof. Up to replacing $(H_i)_{i \in \mathbb{N}}$ by $(H_i)_{i \in S}$, we lighten notation and suppose $S = \mathbb{N}$ or a prefix $\{1, 2, \dots, n\}$ thereof.

Let us write $T_i = \{t_{i,1}, \dots, t_{i,d}\}$ the generating set of H_i . We consider the restricted direct product $B = \prod_{i \geq 1} H_i$. It is a countable, locally finite group, generated by $\{t_{1,1}, \dots, t_{1,d}, t_{2,1}, \dots\} = \{b_1, b_2, \dots\}$. We consider the group W constructed in Section 4, noting that the sequence $n(i)$ may be chosen, by Corollary 4.3, such that W has subexponential growth, and that $n(i)$ depends only on $H_1, H_2, \dots, H_{\lceil i/d \rceil - 1}$.

Consider $[H_i, H_i]$ as a subgroup of W , imbedded as the functions $X \rightarrow H_i \subset B$ supported only at $x_{n(di)}$. This is an imbedding by Lemma 4.1. Denote by $\Psi_i: H_i \rightarrow W$ this imbedding.

Assume that $n(1), \dots, n(di)$ have already been chosen; and note that their choice relies only on H_1, \dots, H_{i-1} . Recall also that $f(x_{n(di-d+j)}) = t_{i,j}$ in the construction of W . We now show that there exist constants K, L independent of H_i such that the imbedding $\Psi_i: [H_i, H_i] \rightarrow W$ is (K, L) -bi-Lipschitz. In other words, independently of i , the distortion of $[H_i, H_i]$ in W is at worst $\rho(t) = tK/L$.

Let $g_1, \dots, g_d \in G$ be such that $x_{n(di-d+j)}g_j = x_{n(di)}$ and the only x_k mapped to another x_ℓ under $g_j'g_j^{-1}$ are $x_{n(di-d+j')}g_j'g_j^{-1} = x_{n(di-d+j)}$; such elements exist because (x_i) is separating. Let L' be an upper bound for the lengths of all g_1, \dots, g_d . This condition ensures that the functions f^{g_1}, \dots, f^{g_d} have disjoint support except at $x_{n(di)}$ or where they coincide.

Here is an explicit way of computing the imbedding $\Psi_i: [H_i, H_i] \rightarrow W$: for $h \in [H_i, H_i]$, write it as a minimal-length balanced word in T_i , and map each letter $t_{i,j}$ to f^{g_j} .

On the one hand, $\|\Psi_i(h)\|_W \leq (2L'+1)\|h\|_{\text{perfect}}$ because each letter gets mapped to a word of length $1+2\|g_j\| \leq 1+2L'$; on the other hand, $\|\Psi_i(h)\|_W \geq \|h\|$, because at most one element of T is contributed by each generator of W . \square

Corollary 6.2. *Let $(G_i)_{i \in \mathbb{N}}$ be a sequence of d -generated finite groups.*

Then there exists a family of groups $(W_S)_{S \subset \mathbb{N}}$ indexed by subsets S of \mathbb{N} , each of subexponential growth, with the following property: for all $s \in S$, there is an imbedding $\Psi_s: G_s \rightarrow W_S$ that is (K, L) -bi-Lipschitz with respect to the word metrics, and such that the constants K, L depend only on $\{G_i: i < s\}$.

Proof. Using Proposition 5.1, imbed each G_i in a $(d+1)$ -generated group H_i in such a manner that the inclusion map $\iota_i: (G_i, d_{G_i}) \rightarrow (H_i, d_{\text{perfect}})$ is $(2, 4)$ -bi-Lipschitz. Apply then Proposition 6.1 to the family $(H_i)_{i \in \mathbb{N}}$. \square

7. IMBEDDINGS INTO METRIC SPACES

Let \mathcal{X} be a metric space. Given a sequence of metric spaces such as $((G_i, d_{G_i}))_{i \in \mathbb{N}}$, we say that it *imbeds coarsely* in \mathcal{X} if there exists an unbounded increasing function ρ and a sequence of 1-Lipschitz imbeddings $(\Phi_i: G_i \rightarrow \mathcal{X})$, each with distortion better than ρ . We are interested in the opposite property:

Definition 7.1. Let \mathcal{X} be a metric space. A sequence of metric spaces $((G_i, d_{G_i}))_{i \in \mathbb{N}}$ *does not imbed coarsely* in \mathcal{X} if the following holds: there exists a constant M such that, if $(\Phi_i: G_i \rightarrow \mathcal{X})$ is a sequence of 1-Lipschitz imbeddings, then, for all $t \in \mathbb{R}$, there are $i \in \mathbb{N}$ and $x, y \in G_i$ with $d(x, y) \geq t$ and $d(\Phi_i(x), \Phi_i(y)) \leq M$.

Theorem 7.2. *Let $(G_i)_{i \in \mathbb{N}}$ be a sequence of d -generated finite groups. Assume furthermore that (G_i) does not imbed coarsely in a metric space \mathcal{X} . Let ρ be any unbounded increasing function $\mathbb{R}_+ \rightarrow \mathbb{R}_+$. Then there exists a finitely generated group W of subexponential growth such that every imbedding of W in \mathcal{X} has distortion worse than ρ .*

Proof. Using Proposition 5.1, imbed each G_i in a group H_i whose derived subgroup, with the perfect metric, contains a bi-Lipschitz copy of G_i . We identify G_i with its image in H_i .

Since the groups G_i do not imbed coarsely in \mathcal{X} , there exists a constant M such that, if $(\Phi_i: G_i \rightarrow \mathcal{X})$ is a sequence of 1-Lipschitz imbeddings then, for all $t \in \mathbb{R}$, there are $i \in \mathbb{N}$ and $x, y \in G_i$ with $d(x, y) \geq t$ and $d(\Phi_i(x), \Phi_i(y)) \leq M$.

The group W will be of the form $W = W_S \times W_{S'}$, for sequences $S = \{s(1), s(3), s(5), \dots\}$ and $S' = \{s(2), s(4), \dots\}$ that we construct iteratively by applying Proposition 6.1 to the family (H_i) . Assume that $s(1), \dots, s(i-1)$ have been constructed. The terms $s = s(i)$ and $s' = s(i+1)$ have not yet been determined, but we already know the constants $K_i, L_i, K_{i+1}, L_{i+1}$ such that the imbedding of G_s into W_S or $W_{S'}$ will be (K_i, L_i) -bi-Lipschitz and the imbedding of $G_{s'}$ into $W_{S'}$ or W_S will be (K_{i+1}, L_{i+1}) -bi-Lipschitz.

We now make use of the unbounded function ρ . Let $t_i \in \mathbb{R}$ be large enough so that $\rho(t_i) > L_{i+1}M$. Since the (G_i) do not imbed coarsely, we can choose s large enough so that there exist $x, y \in G_s$ with $d(x, y) \geq t_i/K_i$ and $d(\Phi_s(x), \Phi_s(y)) \leq M$ in any 1-Lipschitz imbedding Φ_s of G_s into \mathcal{X} . Without loss of generality, the sequences (t_i) and $(s(i))$ are strictly increasing. This determines $s = s(i)$, and finishes the inductive construction of S and S' .

Let us now check that the group W just constructed has the desired property. Let $\Phi: W \rightarrow \mathcal{X}$ be a 1-Lipschitz imbedding. By composing with the imbedding of G_s in W_S or $W_{S'}$, we get for all $s \in S \cup S'$ imbeddings $\Phi_s = \Phi \circ \Psi_s$ of G_s into \mathcal{X} .

Consider $t \in \mathbb{R}_+$, and suppose $t \geq t_1$. Let i be such that $t_{i-1} \leq t < t_i$. Set $s = s(i)$. Following the construction above, the imbedding Φ_s is (K_i, L_i) -Lipschitz, so there are $x, y \in G_s$ with $d(x, y) \geq t_i/K_i$ so $d(\Psi_s(x), \Psi_s(y)) \geq t_i$ while $d(\Phi_s(x), \Phi_s(y)) \leq L_i M$. This proves that the distortion ρ_Φ of Φ satisfies

$$\rho_\Phi(t) \leq \rho_\Phi(t_i) \leq L_i M < \rho(t_{i-1}) \leq \rho(t),$$

so the distortion of W is worse than ρ . \square

We note from the proof that the distortion of a single copy W_S is worse than ρ along an unbounded sequence.

8. SUPEREXPANDERS

We now exhibit a sequence of finite groups (H_i) with particularly bad imbedding properties and fixed-size generating set. We recall the following definition from [12].

A Banach space \mathcal{X} is called of *type* > 1 , or *B-convex*, if there are $n \in \mathbb{N}$ and $\epsilon > 0$ such that no 1-Lipschitz imbedding $(\mathbb{C}^n, \|\cdot\|_1) \rightarrow \mathcal{X}$ with distortion better than $t \mapsto t/(1 + \epsilon)$ exists. For example, Hilbert space is B-convex with $n = 2$ and any $\epsilon > \sqrt{2} - 1$. Lafforgue gives the following construction of expanders:

Proposition 8.1 ([12, Corollaire 0.5]). *There exists a sequence $(H_i)_{i \in \mathbb{N}}$ of finite quotients of a finitely generated group H such that the sequence of quotient Cayley graphs of the H_i does not admit any coarse imbedding into a B-convex Banach space.*

Proof. What Lafforgue shows, actually, is that there exists a constant C such that, for every $i \in \mathbb{N}$ and every 1-Lipschitz map $\Phi: H_i \rightarrow \mathcal{X}$ with 0 mean, one has $\mathbb{E}_{x \in H_i} \|\Phi(x)\|^2 \leq C$. A classical argument (see, e.g., [11, page 600]) implies that there are two points $x, y \in H_i$ with $d(x, y) \geq c \log(\#H_i)$ and $d(\Phi(x), \Phi(y)) \leq \sqrt{2C}$, for a constant c depending only on the number of generators of H . \square

Here is a concrete example: consider a prime power q , the group $H = \mathbf{SL}_3(\mathbb{F}_q[t])$, and its images H_i in $\mathbf{SL}_3(\mathbb{F}_q[t]/(t^i))$. In this situation, we have a few extra, useful properties, which we quote as a

Lemma 8.2. *Additionally, in Proposition 8.1, the group H may be supposed to be perfect.*

Proof. Since $\mathbb{F}_q[t]$ is a Euclidean domain, H is generated by elementary matrices. Furthermore, the classical identities $X_{i,j}(P+Q) = X_{i,j}(P)X_{i,j}(Q)$ and $X_{i,j}(PQ) = [X_{i,k}(P), X_{k,j}(Q)]$ between elementary matrices, when $\{i, j, k\} = \{1, 2, 3\}$, imply that H is generated by $A = \mathbf{SL}_3(\mathbb{F}_q)$ and $B = \langle X_{1,2}(t) \rangle$. Since A is perfect and $B^A = \{1 + tM : M \in M_3(\mathbb{F}_q) \text{ and } \text{tr}(M) = 0\}$ has no A -invariant element, H is also perfect. \square

We fix as generating set $T = A \cup B$, and denote by T_i its natural image in H_i .

Lemma 8.3. *Let H be a finitely generated perfect group, and let $(H_i)_{i \in \mathbb{N}}$ be a family of quotients of H . Then the groups H_i are perfect, all generated by the same number of elements, and the identity map $(H_i, d_{\text{perfect}}) \rightarrow (H_i, d_{H_i})$ is $(J^{-1}, 1)$ -bi-Lipschitz for a constant $J \geq 1$ independent of i .*

Proof. The abelianisations of the H_i are quotients of the abelianisation of H , and therefore are trivial. If $H = \langle T \rangle$ be d -generated, then the groups H_i are naturally d -generated by the images T_i of T .

Represent now each $t \in T$ as a balanced word, and let J be the maximal length of these balanced words. Then

$$\|g\|_{T_i} \leq \|g\|_{\text{perfect}} \leq J\|g\|_T \leq J\|g\|_{T_i} \text{ for all } g \in [H_i, H_i],$$

so the inclusion $([H_i, H_i], d_{\text{perfect}}) \rightarrow (H_i, d_{T_i})$ is $(J^{-1}, 1)$ -bi-Lipschitz. \square

Corollary 8.4. *Let \mathcal{X} be a B -convex Banach space, and let ρ be an unbounded increasing function $\mathbb{R}_+ \rightarrow \mathbb{R}_+$. Then there exists a finitely generated group W of subexponential growth such that every imbedding of W in \mathcal{X} has distortion worse than ρ .*

In particular, let ρ be any unbounded increasing function $\mathbb{R}_+ \rightarrow \mathbb{R}_+$. Then there exists a finitely generated group W of subexponential growth such that every imbedding of W into Hilbert space has distortion worse than ρ .

Proof. Consider the sequence of superexpanders $(H_i)_{i \in \mathbb{N}}$ given by Proposition 8.1. Either imbed them as derived subgroups in finite groups, using Proposition 5.1, or note that that step is unnecessary, thanks to Lemma 8.3.

By Proposition 8.1, there exists a constant M such that if $(\Phi_i : H_i \rightarrow \mathcal{X})$ is a sequence of 1-Lipschitz imbeddings into \mathcal{X} then for every $t \in \mathbb{R}$ and for all j large enough (depending on t) there exist $x, y \in H_j$ with $d(x, y) \geq t$ and $d_{\mathcal{X}}(\Phi_j(x), \Phi_j(y)) \leq M$.

Theorem 7.2 then applies. \square

Remark 8.5. The order of quantifiers can be switched in Corollary 8.4: fixing $n \in \mathbb{N}$ and $\epsilon > 0$, there exists for every unbounded increasing function $\mathbb{R}_+ \rightarrow \mathbb{R}_+$ a group W of subexponential growth with the following property: if \mathcal{X} is any B -convex Banach space not $(1 + \epsilon)$ -isometrically containing $\ell_1(\mathbb{C}^n)$, then every imbedding of

W in \mathcal{X} has distortion worse than ρ . This follows formally from Corollary 8.4 because an ℓ^2 -sum of such Banach spaces is again of the same form.

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